MODULE 3

Vector Differential Calculus

Mathematics-I — MA101

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Department of Mathematics

Syllabus Coverage

Topic
Derivatives of vector valued functions; vector equations of curves; tangents of a curve
Gradient and directional derivative of a scalar field
Divergence of a vector field
Curl of a vector field
Line integrals and double integrals

Chapter 1: Vector-Valued Functions and Parametric Curves

The application of vector calculus to geometry is called differential geometry. Bodies that move in space trace out paths that can be represented by curves C. This motivates the parametric representation of a curve C, using a parameter t (which may denote time or some other quantity).

1.1 Parametric Representation of a Curve

A curve C in space is represented parametrically by the position vector:

$$r(t) = [x(t), y(t), z(t)] = x(t)\, i + y(t)\, j + z(t)\, k$$

Here t is the parameter and x, y, z are the usual Cartesian coordinates. Each value $t = t0$ corresponds to a point of C with position vector r(t0), whose coordinates are x(t0), y(t0), z(t0).

Note: Parametric form (1) treats x, y, z symmetrically as independent variables and induces an orientation on C — the sense of increasing t is the positive sense; the sense of decreasing t is the negative sense.

1.2 Straight Line, Plane, Twisted and Simple Curves

A straight line through (x1, y1, z1) with direction ratios l, m, n has the symmetric (parametric) form:

$$(x - x1)/l = (y - y1)/m = (z - z1)/n$$

Term	Meaning
Plane curve	A curve that lies entirely in a plane in space
Twisted curve	A curve that is not planar
Simple curve	A curve without multiple points (does not intersect or touch itself); e.g. circle, helix
Arc	The portion of a curve between any two of its points

1.3 Tangent to a Curve

The tangent to a simple curve C at a point P is the limiting position of a line L through P and a nearby point Q of C as Q approaches P along C. If P and Q correspond to parameters t and $t + \Delta t$, a vector along L is:

$$(1/\Delta t)[r(t + \Delta t) - r(t)]$$

Letting $\Delta t \to 0$ gives the derivative $r'(t)$, tangent to C at P. The unit tangent vector is:

$$u = (1/|r'|)\, r'$$

The tangent line itself, with parameter w, is given by:

$$q(w) = r + w\, r'$$

This is the sum of the position vector r of P and a multiple of the tangent vector r' of C at P.

Chapter 2: Length of a Curve and the Unit Tangent

2.1 Arc Length

If r(t) has a continuous derivative, the length of the curve C between t = a and t = b — assuming C is rectifiable — is given by the integral:

$$l = \int[a\to b]\ \sqrt{(r' \cdot r')}\ dt$$

Worked Example — dr/ds is a Unit Tangent of C

If s is the arc length of a curve C: r(t), prove that dr/ds is a unit tangent of C.

Solution:

We have $s(t) = \int[a\to t]\ \sqrt{(r' \cdot r')}\ d\tilde{t}$, where $r' = dr/dt$. By the Fundamental Theorem of Calculus:

$$ds/dt = \sqrt{(dr/dt \cdot dr/dt)} = |dr/dt| \quad \ldots (A)$$

By the chain rule, dr/dt = (dr/ds)(ds/dt). Substituting into the dot product $dr/dt \cdot dr/dt$ and using (A):

$$(ds/dt)^2 = |dr/ds|^2 (ds/dt)^2 \implies |dr/ds| = 1$$

Moreover, since $dr/dt = (dr/ds)(ds/dt)$, the vector dr/ds is parallel to dr/dt. Hence dr/ds is a unit vector that is tangent to C — that is, dr/ds is a unit tangent of C.

Chapter 3: Directional Derivatives and Gradient of a Scalar Field

3.1 Gradient of a Scalar Field

Let $\nabla = i\, \partial/\partial x + j\, \partial/\partial y + k\, \partial/\partial z$ be the vector differential (del) operator. The gradient of a scalar function f is defined as:

$$\text{grad } f = \nabla f = (\partial f/\partial x)\, i + (\partial f/\partial y)\, j + (\partial f/\partial z)\, k$$

Worked Example — Gradient of f(x, y, z) = xyz

Find the gradient of $f(x, y, z) = xyz$ at the point P(1, 2, 3).

Solution:

By definition, $\text{grad } f = \partial f/\partial x\, i + \partial f/\partial y\, j + \partial f/\partial z\, k = yz\, i + xz\, j + xy\, k$.

(The value at P is not required further here — the general gradient field is what is used in later examples.)

3.2 Directional Derivative

Let $f(x, y, z)$ be a scalar function. Its directional derivative at P in the direction of a unit vector b is the rate of change of f at P in that direction:

$$D_b f = df/ds = \lim_{s\to 0} [f(Q) - f(P)] / s$$

where Q is a variable point on the ray $C : r(s) = p0 + s\, b$, and s is the distance between P and Q. Using the chain rule for $f(x(s), y(s), z(s))$:

$$D_b f = (\text{grad } f) \cdot dr/ds$$

Since $dr/ds = b$ for the ray $r(s) = p0 + s\, b$, this gives:

$$D_b f = b \cdot (\text{grad } f)$$

More generally, for any vector $a \neq 0$ (not necessarily a unit vector), the directional derivative of f in the direction of a is:

$$D_a f = (1/|a|)\, (a \cdot \text{grad } f)$$

Worked Example — Directional Derivative of f(x, y) = eˣ cos y

Find the directional derivative of $f(x, y) = e^x \cos y$ at the point P(2, π, 0) in the direction of $a = 2i + 3j$.

Solution:

$\text{grad } f = (\partial f/\partial x)\, i + (\partial f/\partial y)\, j = e^x \cos y\, i - e^x \sin y\, j$.

At (2, π, 0): $[\nabla f] = e^2 \cos \pi\, i + (-e^2 \sin \pi)\, j = -e^2\, i$.

$$D_a f = [a/|a|] \cdot \nabla f = [(2i + 3j)/\sqrt{13}] \cdot (-e^2\, i) = -2e^2/\sqrt{13}$$

3.3 Gradient is Normal to a Level Surface

Theorem: Let f be a differentiable scalar function representing a surface S: $f(x, y, z) = c$ (constant). If grad f at a point P of S is not the zero vector, then grad f is a normal vector of S at P.

Sketch of proof: For any curve C: $r(t)$ on S, $f(x(t), y(t), z(t)) = c$, so $df/dt = 0$. By the chain rule this gives $(\text{grad } f) \cdot r' = 0$. Since r' is tangent to S, grad f is perpendicular to every tangent direction at P — hence grad f is normal to S at P.

3.4 Useful Results Involving the Gradient

Identity	Statement
Product rule	$\nabla(fg) = f\,\nabla g + g\,\nabla f$
Laplacian of a product	$\nabla^2(fg) = g\nabla^2 f + f\nabla^2 g + 2(\nabla f \cdot \nabla g)$
Quotient rule	$\nabla(f/g) = (1/g^2)(g\nabla f - f\nabla g)$

Note: These identities are proved by expanding the gradient/Laplacian operator component-wise and applying the product rule of ordinary differentiation to each Cartesian component.

Chapter 4: Divergence of a Vector Field

4.1 Definition

For a vector function $v(x, y, z) = v1\, i + v2\, j + v3\, k$, the divergence is the scalar function:

$$\text{div } v = \nabla \cdot v = \partial v1/\partial x + \partial v2/\partial y + \partial v3/\partial z$$

Worked Example — Divergence of a Given Field

Find the divergence of $v = 2x^2z\, i - xy^2\, j + 3yz^2\, k$.

Solution:

$\text{div } v = \partial/\partial x(2x^2z) + \partial/\partial y(-xy^2) + \partial/\partial z(3yz^2) = 4xz - 2xy + 6yz$.

Worked Example — Divergence of an Exponential Field

Find the divergence of $v = e^{-x}\, i + e^{-x}\, y\, j + 2z \sinh x\, k$.

Solution:

$\text{div } v = \partial/\partial x(e^{-x}) + \partial/\partial y(e^{-x}\, y) + \partial/\partial z(2z \sinh x) = -e^{-x} + e^{-x} + 2 \sinh x = 2 \sinh x = e^x - e^{-x}$.

4.2 Physical Significance

The divergence of a vector field at a point P represents the volume density of outward flux at that point — it measures the net rate at which the field ‘spreads out’ from P. If the flow is incompressible, the divergence vanishes:

$$\text{div } v = 0 \quad \text{(incompressible flow)}$$

4.3 Useful Results Involving Divergence

Identity	Statement
Scalar multiple	div(k v) = k div v, for constant k
Product with a scalar field	div(f v) = f div v + v · ∇f
Divergence of f ∇g	div(f ∇g) = f ∇^2g + (∇f · ∇g)

Practice Problem — Laplacian of xy/z

Find the value of $\nabla^2 f$ if f = xy/z.

Solution:

$\partial f/\partial x = y/z$, $\partial f/\partial y = x/z$, $\partial f/\partial z = -xy/z^2$.

$\partial^2 f/\partial x^2 = 0$, $\partial^2 f/\partial y^2 = 0$, $\partial^2 f/\partial z^2 = 2xy/z^3$.

$$\nabla^2 f = 0 + 0 + 2xy/z^3 = 2xy/z^3$$

Chapter 5: Curl of a Vector Field

5.1 Definition

The curl operator, denoted ∇×, acts on a vector field and returns another vector field. If A = ax i + ay j + az k, then curl A = ∇ × A, best remembered in determinant form:

$$\text{curl } A = i\,(\partial az/\partial y - \partial ay/\partial z) + j\,(\partial ax/\partial z - \partial az/\partial x) + k\,(\partial ay/\partial x - \partial ax/\partial y)$$

Note: If ∇ × A = 0 throughout a region, A is said to be irrotational there.

Worked Example — Curl of a Vector Field

If $A = (y^4 - x^2z^2)\,i + (x^2 + y^2)\,j - x^2yz\,k$, determine curl A at the point (1, 3, −2).

Solution:

Expanding the determinant gives curl $A = i(-x^2z) - j(-2xyz + 2x^2z) + k(2x - 4y^3) = -x^2z\,i - (2x^2z - 2xyz)\,j + (2x - 4y^3)\,k$.

$$\text{At } (1, 3, -2):\ \ \text{curl } A = 2\,i - 8\,j - 106\,k$$

5.2 Physical Significance

The curl of a vector field at a point P gives the (local) angular velocity of the field there. A vector field is said to be irrotational if curl v = 0 at every point; otherwise it is rotational.

5.3 Useful Results on Curl

Identity	Statement
Linearity	curl(u + v) = curl u + curl v
Divergence of a curl	div(curl v) = 0, for any twice-differentiable v

Note: div(curl v) = 0 follows because each term of the expansion involves a mixed second partial derivative that cancels with an identical term of opposite sign, using the equality of mixed partial derivatives ($\partial^2/\partial x\partial y = \partial^2/\partial y\partial x$, etc.).

Chapter 6: Line Integrals

6.1 Introduction

Let C be a curve with parametric representation r(t) = [x(t), y(t), z(t)] = x(t) i + y(t) j + z(t) k, a ≤ t ≤ b. C is called the path of integration; A: r(a) is its initial point and B: r(b) its terminal point. The direction of increasing t is the positive direction on C. If A and B coincide, C is a closed path.

6.2 Physical Applications

Application
Mass of a wire
Centre of mass and moments of inertia of a wire
Work done by a force on an object moving in a vector field
Magnetic field around a conductor (Ampère's Law)
Voltage generated in a loop (Faraday's Law of magnetic induction)

6.3 General Properties of the Line Integral

Property	Statement
Scalar multiple	$\int_C k\,F(r) \cdot dr = k \int_C F(r) \cdot dr$, k constant
Linearity	$\int_C (a_1G_1 + a_2G_2 + \ldots + a_nG_n) \cdot dr = a_1\int_C G_1\cdot dr + a_2\int_C G_2\cdot dr + \ldots + a_n\int_C G_n\cdot dr$
Additivity over sub-paths	$\int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr + \ldots + \int_{C_n} F\cdot dr$, $C = C_1 + C_2 + \ldots + C_n$
Reversal of direction	$\int_{[a\to b]} F\cdot dr = -\int_{[b\to a]} F\cdot dr$

6.4 Line Integral of a Scalar Field

If a scalar field V exists at all points of a curve C, the line integral of V along C from A to B is $\int_C V\,dr$, obtained by expressing V and dr in terms of x, y, z along the parametrisation of C.

Worked Example — Line Integral of a Scalar Field

If $V = xy + y^2z$, evaluate $\int_C V\,dr$ along the curve C: $x = t^2$, $y = 2t$, $z = t + 5$, between A(0, 0, 5) and B(4, 4, 7).

Solution:

Along C, $V = (t^2)(2t) + (4t^2)(t + 5) = 6t^3 + 20t^2$, and dr = 2t dt i + 2 dt j + dt k. A corresponds to t = 0 and B to t = 2, so:

$$\int_C V\,dr = \int_{[0\to2]} (6t^3 + 20t^2)(2t\,i + 2\,j + k)\,dt = (8/15)(444\,i + 290\,j + 145\,k)$$

6.5 Line Integral of a Vector Field

If a vector field $F = F_x\,i + F_y\,j + F_z\,k$ exists at all points of C, its line integral from A to B is:

$$\int_C F \cdot dr = \int_C F_x\,dx + \int_C F_y\,dy + \int_C F_z\,dz$$

Since $F \cdot dr$ is a scalar product, the line integral is a scalar. If F is a force field, $\int_C F \cdot dr$ represents the work done in moving a unit particle along C from A to B.

Worked Example — Line Integral of a Vector Field

If $F = x^2y\,i + xz\,j - 2yz\,k$, evaluate $\int_C F \cdot dr$ between A(0, 0, 0) and B(4, 2, 1) along the curve $x = 4t$, $y = 2t^2$, $z = t^3$.

Solution:

Along C, $F = 32t^4\,i + 4t^4\,j - 4t^5\,k$ and $dr = 4\,dt\,i + 4t\,dt\,j + 3t^2\,dt\,k$, with $t = 0$ at A and $t = 1$ at B. Then:

$$\int_C F \cdot dr = \int_{[0\to1]} (128t^4 + 16t^5 - 12t^7)\,dt = 128/5 + 8/3 - 3/2 = 803/30 \approx 26.77$$

6.6 Path Dependence of a Line Integral

For $F = x^2y\,i + 2yz\,j + 3z^2x\,k$, evaluate $\int_C F \cdot dr$ between A(0, 0, 0) and B(1, 2, 3): (a) along the straight-line segments C1: (0,0,0)→(1,0,0), C2: (1,0,0)→(1,2,0), C3: (1,2,0)→(1,2,3); (b) along the single straight line C4 joining (0,0,0) to (1,2,3).

Solution:

$F \cdot dr = x^2y\,dx + 2yz\,dy + 3z^2x\,dz$. On C1 ($y = z = 0$) and C2 ($x = 1$, $z = 0$) every term vanishes, giving zero on both. On C3 ($x = 1$, $y = 2$): $\int_{C3} F\cdot dr = \int_{[0\to3]} 3z^2\,dz = 27$. Summing, $\int_{(C1+C2+C3)} F \cdot dr = 27$.

On C4: $x = t$, $y = 2t$, $z = 3t$ ($0 \le t \le 1$), so $F = 2t^3\,i + 12t^2\,j + 27t^3\,k$ and $dr = (i + 2j + 3k)\,dt$. Then:

$$\int_{C4} F \cdot dr = \int_{[0\to1]} (2t^3 + 24t^2 + 81t^3)\,dt = 83/4 + 8 = 115/4 = 28.75$$

Note: The value of the line integral (27 via C1+C2+C3, versus 28.75 via C4) depends on the path taken between the same two endpoints — this F is not a conservative (path-independent) field.

6.7 Line Integrals Independent of Path

A line integral $\int_C F(r) \cdot dr = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$ is independent of path in a domain D if, for every pair of endpoints A and B in D, it has the same value along every path in D joining A to B.

A domain D is simply connected if every closed curve in D can be shrunk continuously to a point of D without leaving D.

Theorem	Statement
Theorem 3 (necessary condition)	If F_1, F_2, F_3 are continuous with continuous first partials in D and $\int_C(F_1dx+F_2dy+F_3dz)$ is independent of path in D, then curl F = 0 in D — i.e. $\partial F_3/\partial y = \partial F_2/\partial z$, $\partial F_1/\partial z = \partial F_3/\partial x$, $\partial F_2/\partial x = \partial F_1/\partial y$.
Theorem 4 (sufficient condition)	If F_1, F_2, F_3 are continuous with continuous first partials in D, curl F = 0 in D, and D is simply connected, then $\int_C(F_1dx+F_2dy+F_3dz)$ is

Theorem	Statement
	independent of path in D.

Chapter 7: Double Integrals

7.1 Definition

Let $f(x, y)$ be a function defined on a region R of the plane. Subdivide R into small rectangular subregions, the k-th of dimensions Δx_k, Δy_k and area $\Delta A_k = \Delta x_k \cdot \Delta y_k$. Choosing an arbitrary point (x_k, y_k) in each subregion, form the Riemann sum:

$$J_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

If f is continuous on R and R is bounded by finitely many smooth curves, the sequence (J_1, J_2, ...) converges to a limit independent of the choice of subdivisions and sample points. This limit is the double integral of f over R:

$$\iint_R f(x, y)\, dx\, dy \quad \text{or} \quad \iint_R f(x, y)\, dA$$

7.2 Applications of Double Integrals

Application
Area of a plane region
Volume beneath a surface (e.g. an elliptic paraboloid)
Average value of a function over a region
Surface integrals
Mass, centre of mass, and moments of a lamina

7.3 Properties of the Double Integral

Property	Statement
Linearity (sum)	$\iint_A [f(x,y) + g(x,y)]\, dA = \iint_A f\, dA + \iint_A g\, dA$
Linearity (scalar)	$\iint_A c\, f(x,y)\, dA = c \iint_A f(x,y)\, dA$
Comparison	If $f(x,y) \geq g(x,y)$ on R, then $\iint_A f\, dA \geq \iint_A g\, dA$

Worked Example — Volume Beneath a Surface

Find the volume of the region beneath $z = 4x^2 + 9y^2$ over the rectangle with vertices (0,0), (3,0), (3,2), (0,2).

Solution:

The volume beneath $z = f(x, y)$ (> 0) and above region R in the xy-plane is $V = \iint_R f(x, y)\, dx\, dy$. Integrating first over x from 0 to 3:

$$V = \int_{y=0}^{2} \int_{x=0}^{3} (4x^2 + 9y^2)\,dx\,dy = \int_{y=0}^{2} (36 + 27y^2)\,dy = [36y + 9y^3] \text{ from 0 to 2}$$

$$V = 72 + 72 = 144$$

Quick Reference — All Key Formulae (Module 3)

Curves and Tangents

$r(t) = x(t)\, i + y(t)\, j + z(t)\, k$; tangent line: $q(w) = r + w\, r'$; unit tangent: $u = r'/|r'|$

Arc Length

$l = \int_a^b \sqrt{r' \cdot r'}\, dt$; dr/ds is a unit tangent of C

Gradient and Directional Derivative

$\text{grad } f = \nabla f = f_x\, i + f_y\, j + f_z\, k$; $D_a f = (1/|a|)(a \cdot \nabla f)$; ∇f is normal to $f = c$

Divergence

$\text{div } v = \nabla \cdot v = \partial v_1/\partial x + \partial v_2/\partial y + \partial v_3/\partial z$; $\text{div } v = 0$ for incompressible flow

Curl

$\text{curl } A = \nabla \times A$ (determinant form); $\text{curl } v = 0 \Longleftrightarrow v$ irrotational; $\text{div}(\text{curl } v) = 0$

Line Integrals

$\int_C F \cdot dr = \int_C F_x\, dx + F_y\, dy + F_z\, dz$; independent of path in simply connected $D \Longleftrightarrow \text{curl } F = 0$

Double Integrals

$\iint_R f(x, y)\, dA = \lim \Sigma f(x_k, y_k)\, \Delta A_k$; used for area, volume, mass, and averages

— End of Module 3 Notes —